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FACTORIZATION FOR NON-NEVANLINNA CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT

A generalization of the Blaschke product is constructed. This product enables one to factor out the zeros of the members of certain non-Nevanlinna classes of functions analytic in the unit disc, so that the remaining (non-vanishing) functions still belong to the same class. This is done for the classes $\mathcal{A}^{-n}$ $(0 < n < \infty)$ and $\mathcal{B}^{-n}$ $(0 < n < 2)$ defined as follows: $f \in \mathcal{A}^{-n}$ iff $|f(z)| \leqq C_f(1-|z|)^{-n}$, $f \in \mathcal{B}^{-n}$ iff $|f(z)| \leqq \exp\{C_f(1-|z|)^{-n}\}$, where C_f depends on f.

1. Introduction: generalized Blaschke products

Non-Nevanlinna classes of analytic functions have been a subject of recent interest, in particular with respect to questions of zero sets and factorization. One example of such a class is given by $\mathcal{A}^{-n}$ $(0 < n < \infty)$, the class of all functions analytic in the unit disc satisfying

$$|f(z)| \leqq C(1-|z|)^{-n} \qquad (|z|<1)$$

for some $C > 0$. If we equip $\mathcal{A}^{-n}$ with the norm

$$\|f\|_{-n} = \sup_{|z|<1} (1-|z|)^n |f(z)|,$$

then it becomes a Banach space. These spaces are closely related to the Bergman A^p spaces $(0 < p < \infty)$ defined by

$$f \in A^p \Leftrightarrow \int\int_{|z|<1} |f(z)|^p \, dxdy < \infty, f \quad \text{analytic in} \quad |z|<1.$$

For $p \geqq 1$, A^p is also a Banach space under the obvious norm, and for all positive p and n, the following inclusions hold:

$$\mathcal{A}^{-n} \subset A^{(1/n)+\varepsilon}, \forall \varepsilon > 0; \qquad A^p \subset \mathcal{A}^{-2/p}.$$

(The second inclusion follows from the fact that $|f(z)|^p$ is subharmonic.)

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The classes $\mathcal{A}^{-n}$, and their union $\mathcal{A}^{-\infty} = \bigcup_{\{n>0\}} \mathcal{A}^{-n}$, were studied extensively by B. Korenblum [3]. In dealing with the factorization of $\mathcal{A}^{-\infty}$, he constructed a generalization of the Blaschke product [3, sec. 6.1], and showed that if $\tilde{B}(z)$ is the generalized Blaschke product for any subset of the zero set of $f \in \mathcal{A}^{-n}$, then $g \in \mathcal{A}^{-4n-\varepsilon}(\forall \varepsilon > 0)$, where $g = f/\tilde{B}$.

In his study of A^p zero sets, C. Horowitz [2, sec. 7] defined a different generalization of the Blaschke product, namely,

$$h(z) = \prod_{k=1}^{\infty} B_{z_k}(z)(2 - B_{z_k}(z)), \tag{1}$$

where $B_a(z)$ is the usual Blaschke factor:

$$B_a(z) = \frac{|a|}{a}\frac{a-z}{1-\bar{a}z} \qquad (0<|a|<1);\ B_0(z) = z. \tag{2}$$

Horowitz proved that if $\{z_k\}$ is any subset of the zero set of $f \in A^p$, and $h(z)$ is as in (1), then f/h also belongs to the same A^p. (As we shall soon see, Horowitz's product does the same job for $\mathcal{A}^{-n}$ spaces.)

We now turn to functions characterized by an exponential growth rate. For $0 < n < \infty$, let $\mathcal{B}^{-n}$ denote the algebra of all functions analytic in $|z| < 1$, which satisfy

$$|f(z)| \leqq \exp\left(\frac{C}{(1-|z|)^n}\right) \qquad (|z|<1)$$

for some $C > 0$.

H. S. Shapiro and A. L. Shields [4] obtained the following interesting result: *If $f \in \mathcal{B}^{-n}$, $n < 1$, then those zeros $\{z_k\}$ of f which lie on a single radius satisfy the Blaschke condition* $\Sigma(1-|z_k|) < \infty$.

Again, we have a related hierarchy of algebras B^p $(0 < p < \infty)$, defined as follows:

$$f \in B^p \Leftrightarrow \int\int_{|z|<1} (\log^+|f(z)|)^p\,dxdy < \infty, \qquad f \quad \text{analytic in} \quad |z|<1.$$

Beller [1] studied the zero sets of B^p functions. $\mathcal{B}^{-n}$ and B^p are related by the following inclusions:

$$\mathcal{B}^{-n} \subset B^{(1/n)+\varepsilon},\ \forall \varepsilon > 0\ (0<n<\infty);\ B^p \subset \mathcal{B}^{-2/p} \qquad (1 \leqq p < \infty).$$

As far as $\mathcal{B}^{-n}$ classes are concerned, the utility of Horowitz's (or Korenblum's) product is limited by the fact that it converges only if $\Sigma(1-|z_k|)^2 < \infty$, while for $f \in \mathcal{B}^{-n}$, $n \geqq 1$, it is possible for $\Sigma(1-|z_k|)^2$ to diverge, as we shall see in the sequel. In order to overcome this difficulty, we define, for $m > 0$, the following product:

$$h_m(z) = \prod_{k=1}^{\infty} (1 - \{1 - B_{z_k}(z)\}^m), \tag{3}$$

where $B_{z_k}(z)$ is the Blaschke factor as in (2). Since $w = 1 - B_{z_k}(z)$ lies within the disc $|w - 1| < 1$ for $|z| < 1$, we are able to take the principal branch of $\{1 - B_{z_k}(z)\}^m$ in case m is not an integer.

In view of the inequality

$$|1 - B_{z_k}(z)| \leqq \frac{2}{1-r}(1 - |z_k|) \qquad (|z| \leqq r), \tag{4}$$

we conclude that the product (3) converges if $\Sigma(1 - |z_k|)^m < \infty$, and, in that case, defines a function analytic in the unit disc. Note that $h_1(z)$ is just the classical Blaschke product, while $h_2(z)$ is the Horowitz–Blaschke product (1).

It is obvious that $h_m(z)$ vanishes at all the z_k. For $m \leqq 6$, it has no other zeros; this is a consequence of the following lemma:

LEMMA 1. *If* $0 < m \leqq 6$ *and* $0 < |w| < 1$, *then* $(1 - w)^m \neq 1$.

PROOF. We may assume, without loss of generality, that $0 \leqq \arg(1 - w) < \pi/2$. If $(1 - w)^m$ were equal to 1, then we would have $|1 - w| = 1$, which implies that $0 < \arg(1 - w) < \pi/3$. Thus, $0 < \arg(1 - w)^m < m\pi/3 \leqq 2\pi$, a contradiction. ■

It is easily seen that Lemma 1 no longer holds when $m > 6$, and thus, for such m, $h_m(z)$ cannot be used for any kind of factorization. For our specific purpose, we will need an inequality which holds only for $m \leqq 3$:

LEMMA 2. *If* $0 < m \leqq 3$ *and* $|w| < 1$, *then*

$$|1 - (1 - w)^m| \geqq 1 - (1 - |w|)^m. \tag{5}$$

PROOF. Let us write $1 - w = re^{i\theta}$. We may again assume that $0 \leqq \theta < \pi/2$. Now $\arg(1 - w)^m = m\theta < 3\pi/2$. Thus, if $m\theta \geqq \pi/2$, the left-hand side of (5) is greater than 1, and the inequality certainly holds. Consequently, we may assume that $m\theta < \pi/2$.

Now inequality (5) is equivalent to $F(\theta) \geqq 0$, $(0 \leqq \theta < \pi/2)$, where

$$F(\theta) = r^{2m} - 2r^m \cos m\theta - (1 - \{1 + r^2 - 2r\cos\theta\}^{\frac{1}{2}})^{2m} + 2(1 - \{1 + r^2 - 2r\cos\theta\}^{\frac{1}{2}})^m.$$

We have $F(0) = 0$, and if one keeps in mind that $m\theta < \pi/2$, a simple calculation shows that $F'(\theta) > 0$ for $0 < \theta < \pi/2$. ■

In the next section, we will use our product to factor out the zeros of $\mathcal{A}^{-n}$ functions in such a way that the quotient remains in $\mathcal{A}^{-n}$; in Section 3, the same will be done for $\mathcal{B}^{-n}$ functions with $n < 2$. (We are unable to extend the $\mathcal{B}^{-n}$ result to $n \geqq 2$ because, in that case, inequality (5) breaks down.) As for the growth rate of $h_m(z)$ itself, a straightforward estimate based on (4) yields an upper bound on the growth, namely

$$|h_m(z)| < \exp\left(\frac{2^m K}{(1-|z|)^m}\right),$$

where $K = \sum_{k=1}^{\infty}(1-|z_k|)^m$. What yet remains to be clarified is the precise growth rate for given $\{z_k\}$.

2. Factorization of $\mathcal{A}^{-n}$ functions

THEOREM 1. *Let $\{z_k\}$ be any subset of the zero set of $f \in \mathcal{A}^{-n}$ $(0 < n < \infty)$, and for any $1 < m \leqq 3$, let $h_m(z)$ be the generalized Blaschke product for $\{z_k\}$, as in (3). Then*

(i) *$h_m(z)$ is analytic in $|z| < 1$, with zeros precisely at $\{z_k\}$.*

(ii) *Set $g = f/h_m$. There exists a number $A(n, m)$, depending only on n and m, such that*

$$\|g\|_{-n} \leqq A(n, m)\|f\|_{-n}.$$

COROLLARY 1. *Every subset of an $\mathcal{A}^{-n}$ zero set is an $\mathcal{A}^{-n}$ zero set.*

Before proving Theorem 1, we must establish some preliminary facts:

PROPOSITION 1. *If $f \in \mathcal{A}^{-n}$ $(0 < n < \infty)$, with zero set $\{a_k\}$, and if $f(0) \neq 0$, then*

$$\prod_{k=1}^{N} |a_k|^{-1} \leqq \frac{e^{n+1}C}{|f(0)|} N^n \qquad (N = 1, 2, 3, \cdots), \tag{6}$$

where $C = \|f\|_{-n}$.

REMARK 1. Inequality (6) implies the well-known conclusion that $\Sigma(1-|a_k|)^{1+\varepsilon} < \infty$ for all $\varepsilon > 0$.

REMARK 2. Using the same construction as Horowitz [2, Sec. 4], it is not difficult to see that the exponent of N in Proposition 1 is the best possible, and that for $0 < n < p < \infty$, there exist $\mathcal{A}^{-p}$ zero sets which are not $\mathcal{A}^{-n}$ zero sets.

PROOF OF PROPOSITION 1. First let us assume that $\{a_k\}$ is an *ordered* zero sequence of f, i.e., $0<|a_1|\leqq|a_2|\leqq\cdots$. From Jensen's formula we can conclude (cf. [2, Sec. 3]) that for $0\leqq r<1$, and for all positive integers N,

$$|f(0)|\prod_{k=1}^{\infty}\frac{r}{|a_k|}\leqq\exp\left((2\pi)^{-1}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta\right). \tag{7}$$

Since $\|f\|_{-n}=C$, the right hand side of (7) is at most $C/(1-r)^n$, so that

$$\prod_{k=1}^{N}|a_k|^{-1}\leqq C|f(0)|^{-1}r^{-N}(1-r)^{-n}\qquad(0<r<1;N=1,2,3,\cdots).$$

For each N, choose $r=N/(N+n)$. Elementary estimates then yield (6). If we arbitrarily reorder the zero sequence of f, then (6) holds *a fortiori*. ■

LEMMA 3. *Let N be a natural number, and let $F\in C^1[0,1]$ such that $F(x)$ is strictly increasing and $xF'(x)/F(x)$ is strictly decreasing. If $0<a_1\leqq a_2\leqq\cdots\leqq a_N\leqq 1$ and $0<b_1\leqq b_2\leqq\cdots\leqq b_N<1$, and if*

$$\prod_{k=1}^{n}a_k\geqq\prod_{k=1}^{n}b_k\qquad(n=1,2,\cdots,N),$$

then

$$\prod_{n=1}^{N}F(a_n)\geqq\prod_{n=1}^{N}F(b_n).$$

PROOF. Horowitz [2a, lemma 7.12] proved a special case of this lemma, in which $F(x)=x(2-x)$, $b_1=1/C$, and $b_k=((k-1)/k)^{\beta}$, $(k\geqq 2)$, for some $\beta>0$. The proof of our generalized lemma proceeds exactly as his proof, to which the reader is referred. The only detail to be checked is that

$$G(\delta)\equiv F(a_ne^{-\delta})\{F(a_{n+1}e^{\delta/k})\}^k$$

is a strictly decreasing function of δ for $\delta>0$, where $a_n\leqq a_{n+1}<1$, and k is any natural number. But an elementary calculation shows that $G'(\delta)<0$ iff $xF'(x)/F(x)$ is strictly decreasing. ■

LEMMA 4. *Let n and C be positive numbers, and let $m>1$. Let $0<a_1\leqq a_2\leqq\cdots\leqq 1$. If*

$$\prod_{k=1}^{N}a_k^{-1}\leqq CN^n$$

for all natural N, then

$$\prod_{k=1}^{\infty}\{1-(1-a_k)^m\}^{-1}\leqq CB(n,m), \tag{8}$$

where $B(n,m)$ depends only on n and m.

PROOF. Let us first assume that $C \geqq 2^n$. Then we can apply Lemma 3, with $F(x) = 1-(1-x)^m$, $b_1 = 1/C$, $b_k = (1-1/k)^n$ $(k = 2,3,\cdots)$, from which we can conclude that

$$\prod_{k=1}^{\infty} \{1-(1-a_k)^m\}^{-1} \leqq \prod_{k=1}^{\infty} \{1-(1-b_k)^m\}^{-1}. \tag{9}$$

Now set

$$N = \max([1/(2n)], [2^{1/m}2n]), \tag{10}$$

where $[\cdot]$ is the greatest-integer function. We have

$$\prod_{k=1}^{N} \{1-(1-b_k)^m\}^{-1} \leqq \prod_{k=1}^{N} b_k^{-1} = CN^n. \tag{11}$$

Now if $n > 1$, we have $b_k > 1 - n/k$, and if $0 < n < 1$, $b_k > 1-(n/k) - 1/(2k^2-2k)$. Thus, in any case, $b_k > 1 - 2n/k$ provided that $k \geqq 1 + 1/(2n)$. If, in addition, $k \geqq 2^{1/m}2n$, then we have $(2n/k)^m < \frac{1}{2}$ and therefore

$$-\log(1-(1-b_k)^m) \leqq -\log(1-(2n/k)^m) < \frac{3}{2}\left(\frac{2n}{k}\right)^m.$$

Thus, taking (10) into account, we have

$$\log \prod_{k=N+1}^{\infty} \{1-(1-b_k)^m\}^{-1} < \sum_{k=N+1}^{\infty} \frac{3}{2}\left(\frac{2n}{k}\right)^m < \frac{3(2n)^m}{2(m-1)N^{m-1}}.$$

Combining this with (9) and (11) yields

$$\prod_{k=1}^{\infty} \{1-(1-a_k)^m\}^{-1} \leqq CD(n,m)$$

where $D(n,m) = N^n \exp(3(2n)^m/\{2(m-1)N^{m-1}\})$, and N is given by (10). If $c < 2^n$, then (8) remains correct if we write $B(n,m) = 2^n D(n,m)$. ■

PROOF OF THEOREM 1. Part (i) is true in view of Lemma 1 and the first remark to Proposition 1. To prove (ii), set $c_w(z) = (w-z)/(1-\bar{w}z)$, $(|w|<1)$, and $f_w(z) = f(c_w(z))$. Writing $C = \|f\|_{-n}$, we have

$$|f_w(z)| \leqq \frac{C}{\left(1 - \left|\dfrac{w-z}{1-\bar{w}z}\right|\right)^n} < \frac{2^n C(1-|w|)^{-n}}{(1-|z|)^n}, \tag{12}$$

the last inequality following from an elementary estimate.

Let $\{a_k\}$ be the *complete* zero set of $f(z)$. Then $\{c_w(a_k)\}$ is the complete zero set of $f_w(z)$. Now let $\{z_k\}$ be an arbitrary subset of the zero set of f. Writing $g = f/h_m$, and taking into account Lemma 2, together with the fact that

$$|B_{a_k}(w)| = |c_{a_k}(w)| = |c_w(a_k)|,$$

we have

$$
\begin{aligned}
|g(w)| &\leqq |f(w)| \prod_{k=1}^{\infty} (1-\{1-|B_{z_k}(w)|\}^m)^{-1} \\
&\leqq |f(w)| \prod_{k=1}^{\infty} (1-\{1-|c_w(a_k)|\}^m)^{-1}.
\end{aligned} \tag{13}
$$

Now assume that $f_w(0) \equiv f(w) \neq 0$. Applying Proposition 1 to f_w, in light of (12), we conclude that

$$\prod_{k=1}^{N} |c_w(a_k)|^{-1} \leqq \frac{e(2e)^n C(1-|w|)^{-n}}{|f(w)|} N^n \qquad (N = 1,2,3,\cdots). \tag{14}$$

Lemma 4, applied to (14), and combined with (13), finally yields

$$|g(w)| \leqq CA(n,m)(1-|w|)^{-n}.$$

By continuity, this inequality holds also at those w for which $f(w) = 0$. ■

3. Factorization of $\mathscr{B}^{-n}$ functions

We begin with an estimate on the moduli of the zeros of $\mathscr{B}^{-n}$ functions.

THEOREM 2. *Let $f \in \mathscr{B}^{-n}$ $(0 < n < \infty)$ be such that*

$$|f(z)| \leqq \exp\left(\frac{C}{(1-|z|)^n}\right) \qquad (|z| < 1). \tag{15}$$

If $\{a_k\}$ is its zero set, and if $f(0) \neq 0$, then

$$\prod_{k=1}^{N} |a_k|^{-1} \leqq |f(0)|^{-1} e^{F(n)C} \exp\{E(n) C^{1/(n+1)} N^{n/(n+1)}\} \qquad (N = 1,2,\cdots), \tag{16}$$

where $E(n)$ and $F(n)$ depend only on n.

PROOF. First assume that $\{a_k\}$ is an ordered zero sequence. From Jensen's formula and (15) we conclude that

$$\sum_{k=1}^{N} \log \frac{1}{|a_k|} \leqq \frac{C}{(1-r)^n} - N \log r - \log |f(0)|$$

for all $0 < r < 1$, and for $N = 1,2,3,\cdots$.

For $N \geqq (1/3)nC2^{n+1}$, we choose $r = 1-(2nC/3N)^{1/(n+1)}$. For such N we have $r \geqq \frac{1}{2}$, and therefore $-\log r \leqq (3/2)(1-r)$ holds. Substitution into (17) yields

$$\sum_{k=1}^{N} \log \frac{1}{|a_k|} \leqq E(n)C^{1/(n+1)}N^{n/(n+1)} - \log|f(0)| \qquad (N \geqq (1/3)n2^{n+1}C), \tag{18}$$

where $E(n) = (2n/3)^{1/(n+1)} + (2n/3)^{-n/(n+1)}$. For $N < (1/3)n2^{n+1}C$,

$$\prod_{k=1}^{N} |a_k|^{-1} \leqq \prod_{k=1}^{N^*} |a_k|^{-1} \leqq \frac{e^{F(n)C}}{|f(0)|}, \tag{19}$$

where $N^* = [2^{n+1}nC/3]$, and $F(n) = 2^{n(n+2)/(n+1)}(n/3)^{n/(n+2)}E(n)$. Inequalities (18) and (19) together yield (16) for all N. If $\{a_k\}$ is arbitrarily reordered, then (16) holds *a fortiori*. ■

REMARK. If $f(z)$ has a zero of order p at $z = 0$, then one needs only to divide by z^p before applying Theorem 2.

COROLLARY 2. *If $f \in \mathcal{B}^{-n}$ $(0 < n < \infty)$, with zero set $\{a_k\}$, then for all $\varepsilon > 0$,*

$$\sum_{k=1}^{\infty} (1-|a_k|)^{n+1+\varepsilon} < \infty. \tag{20}$$

PROOF. Let $\{a_k\}$ be ordered. It follows from Theorem 2 that $\Sigma_{k=1}^{N}(1-|a_k|) = O(N^{n/(n+1)})$. Since $\{|a_k|\}$ is non-decreasing, we have $N(1-|a_N|) \leqq \Sigma_{k=1}^{N}(1-|a_k|)$, and therefore $1-|a_k| = O(k^{-1/(n+1)})$, which implies (20). ■

REMARK. With the aid of a Horowitz-type construction used by Beller in [1, p. 79], it is not difficult to show that the exponent of N in Theorem 2 is sharp. Consequently, the $n+1$ appearing in the exponent of (20) is best possible. In the same way, for $n < p$, one can construct $\mathcal{B}^{-p}$ zero sets which are not $\mathcal{B}^{-n}$ zero sets.

LEMMA 5. *Let $0 < n < \infty$, and $m > n+1$. Set $\beta = n/(n+1)$. There exists a number $G(n,m)$, depending only on n and m, such that for $b \geqq G(n,m)$ and $a \geqq \exp(-b(2-2^\beta))$, the following implication holds:*

If $0 < a_1 \leqq a_2 \leqq \cdots \leqq 1$, and

$$\prod_{k=1}^{N} a_k^{-1} \leqq a\exp(bN^\beta) \qquad (N = 1,2,3,\cdots)$$

then

$$\prod_{k=1}^{\infty} \{1-(1-a_k)^m\}^{-1} \leqq a\exp(3b^{n+1}). \tag{21}$$

PROOF. After applying Lemma 3 to the case where $F(x) = 1-(1-x)^m$ and

$$b_1 = e^{-b}/a,\ b_k = \exp\{-b(k^\beta-(k-1)^\beta)\} \qquad (k = 2,3,\cdots),$$

we conclude that

$$\prod_{k=1}^{\infty}\{1-(1-a_k)^m\}^{-1} \leqq \prod_{k=1}^{\infty}\{1-(1-b_k)^m\}^{-1}. \tag{22}$$

Thus, we must estimate the right hand side of (22). Setting $N = [b^{n+1}]$, we have

$$\prod_{k=1}^{N}\{1-(1-b_k)^m\}^{-1} \leqq \prod_{k=1}^{N} b_k^{-1} = a\exp(bN^\beta) \leqq a\exp(b^{n+1}). \tag{23}$$

In light of the elementary inequality

$$k^\beta-(k-1)^\beta \leqq k^{\beta-1}\ (0<\beta<1;\ k = 1,2,\cdots)$$

we have

$$\log \prod_{k=N+1}^{\infty}\{1-(1-b_k)^m\}^{-1} < -\sum_{k=N+1}^{\infty}\log(1-\{1-\exp(-bk^{\beta-1})\}^m). \tag{24}$$

Now there exists a number K_m, $0<K_m\leqq\infty$, such that

$$-\log(1-(1-e^{-x})^m) < x^m \qquad (0<x<K_m).$$

Indeed, since $m>1$, we have

$$-\log(1-(1-e^{-x})^m) = x^m - \tfrac{1}{2}mx^{m+1} + \text{higher powers of } x.$$

Thus, if $b \geqq K_m^{-1/n}$, this insures that for $k \geqq N+1$, we have $bk^{\beta-1}<K_m$, and therefore

$$\begin{aligned} -\sum_{k=N+1}^{\infty}\log(1-\{1-\exp(-bk^{\beta-1})\}^m) &< \sum_{k=N+1}^{\infty} b^m k^{-m/(n+1)} \\ &< b^m N^{-(m-n-1)/(n+1)} < b^m(b^{n+1}-1)^{-(m-n-1)/(n+1)}. \end{aligned} \tag{25}$$

Now

$$b^m(b^{n+1}-1)^{-(m-n-1)/(n+1)} \leqq 2b^{n+1} \tag{26}$$

provided

$$b \geqq H(n,m) \equiv (1-2^{-(n+1)/(m-n-1)})^{-1/(n+1)}.$$

Thus, if we set $G(n,m) = \max(K_m^{-1/n}, H(n,m))$, then (22), (23), (24), (25), and (26) together yield (21). ■

We are now able to prove the main theorem:

THEOREM 3. *Let $f \in \mathcal{B}^{-n}$ $(0 < n < 2)$, let m be such that $n+1 < m < 3$, and let $\{z_k\}$ be any subset of the zero set of f. If $h_m(z)$ is the generalized Blaschke product for $\{z_k\}$, as in* (3), *then*

(i) *$h_m(z)$ is analytic in $|z| < 1$ with zeros precisely at $\{z_k\}$.*

(ii) *$f/h_m \equiv g \in \mathcal{B}^{-n}$.*

COROLLARY 3. *For $n < 2$, every subset of a $\mathcal{B}^{-n}$ zero set is a $\mathcal{B}^{-n}$ zero set.*

PROOF OF THEOREM 3. Part (i) follows immediately from Corollary 2. To prove (ii), we use the same notation as in the proof of Theorem 1. If

$$|f(z)| \leqq \exp\left(\frac{C}{(1-|z|)^n}\right) \qquad (|z|<1),$$

then, analogous to inequality (12), we have, for any $|w|<1$,

$$|f_w(z)| \leqq \exp\left(\frac{2^n C(1-|w|)^{-n}}{(1-|z|)^n}\right) \qquad (|z|<1). \tag{27}$$

Assume, again, that $f(w) \neq 0$. Keeping (27) in mind, Theorem 2, applied to f_w, yields

$$\prod_{k=1}^{N} \frac{1}{|c_w(a_k)|} \leqq \frac{1}{|f(0)|} \exp\left(\frac{F^*(n)C}{(1-|w|)^n}\right) \exp\left(\frac{E^*(n)C^{1/(n+1)}}{(1-|w|)^n} N^{n/(n+1)}\right), \tag{28}$$

for $N = 1, 2, 3, \cdots$, where $F^*(n) = 2^n \max(1, F(n))$, and $E^*(n) = E(n)2^{n/(n+1)}$. We now apply Lemma 5 to (28) with

$$a = \frac{1}{|f(w)|} \exp\left(\frac{F^*(n)C}{(1-|w|)^n}\right); \qquad b = \frac{E^*(n)C^{1/(n+1)}}{(1-|w|)^{n/(n+1)}}.$$

(Note that (27), for $z=0$, together with the fact that $F^*(n) \geqq 2^n$, implies that $a \geqq \exp(-b(2-2^\beta))$, as required.) The condition $b \geqq G(n, m)$ in Lemma 5 is equivalent to $|w| \geqq 1 - J(n, m)$, where $J = (E^*/G)^{(n+1)/n} C^{1/n}$. Thus, for such w, Lemma 5, combined with (13), yields

$$|g(w)| \leqq \exp\left(\frac{K(n)C}{(1-|w|)^n}\right),$$

where $K(n) = F^*(n) + 3\{E^*(n)\}^{n+1}$. Now set $L = \sup_{\{|w| \leq 1-J\}} |g(w)|$ and $M = \max(e, L)$. Then for all w (such that $f(w) \neq 0$) we have

$$|g(w)| \leqq \exp\left(\frac{K(n)C\log M}{(1-|w|)^n}\right).$$

By continuity, this inequality holds also for those w which are zeros of f. ■

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